Iterative Krylov Subspace Methods for Sparse Reconstruction

James Nagy

Mathematics and Computer Science
Emory University
Atlanta, GA USA

Joint work with Silvia Gazzola
University of Padova, Italy

Outline

1. Ill-posed inverse problems, regularization, preconditioning
2. Related previous work
3. Our approach to solve the problem
   - First Example
   - Second Example
   - Comparison with other methods
4. Concluding Remarks
5. A Bob Plemmons Story

Linear Ill-Posed Inverse Problem

Consider general problem

$$b = Ax + \boldsymbol{\eta}$$

where

- b is known vector (measured data)
- x is unknown vector (want to find this)
- $\boldsymbol{\eta}$ is unknown vector (noise)
- A is large, ill-conditioned matrix, and generally
  - large singular values $\leftrightarrow$ low frequency singular vectors
  - small singular values $\leftrightarrow$ high frequency singular vectors

Linear Ill-Posed Inverse Problem

Consider general problem

$$b = Ax + \boldsymbol{\eta}$$

where

- b is known vector (measured data)
- x is unknown vector (want to find this)
- $\boldsymbol{\eta}$ is unknown vector (noise)
- A is large, ill-conditioned matrix, and generally
  - large singular values $\leftrightarrow$ low frequency singular vectors
  - small singular values $\leftrightarrow$ high frequency singular vectors

ignore noise, and "solve" $Ax = b \quad \Rightarrow \quad A^{-1}b = x + A^{-1}\boldsymbol{\eta}$

Linear Ill-Posed Inverse Problem

Consider general problem

$$b = Ax + \boldsymbol{\eta}$$

where

- b is known vector (measured data)
- x is unknown vector (want to find this)
- $\boldsymbol{\eta}$ is unknown vector (noise)
- A is large, ill-conditioned matrix, and generally
  - large singular values $\leftrightarrow$ low frequency singular vectors
  - small singular values $\leftrightarrow$ high frequency singular vectors

ignore noise, and "solve" $Ax = b \quad \Rightarrow \quad A^{-1}b = x + A^{-1}\boldsymbol{\eta} \not\approx x$

Linear Ill-Posed Inverse Problem

Consider general problem

$$b = Ax + \boldsymbol{\eta}$$

where

- b is known vector (measured data)
- x is unknown vector (want to find this)
- $\boldsymbol{\eta}$ is unknown vector (noise)
- A is large, ill-conditioned matrix, and generally
  - large singular values $\leftrightarrow$ low frequency singular vectors
  - small singular values $\leftrightarrow$ high frequency singular vectors

ignore noise, and "solve" $Ax = b \quad \Rightarrow \quad A^{-1}b = x + A^{-1}\boldsymbol{\eta} \not\approx x$

- inverting smallest singular values amplifies noise

Regularization for Ill-Posed Inverse Problems

Solutions to these problems usually formulated as:

$$\min_{x} \mathcal{L}(x) + \lambda \mathcal{R}(x)$$

where

- $\mathcal{L}$ is a goodness of fit function (e.g., negative log likelihood)
- $\mathcal{R}$ is regularization (reconstruct high and damp low frequencies)
- λ is regularization parameter

Regularization for Ill-Posed Inverse Problems

Solutions to these problems usually formulated as:

$$\min_x \mathcal{L}(x) + \lambda \mathcal{R}(x)$$

where

- $\mathcal{L}$ is a goodness of fit function (e.g., negative log likelihood)
- $\mathcal{R}$ is regularization (reconstruct high and damp low frequencies)
- λ is regularization parameter

Example: Standard Tikhonov regularization:

$$\min_x \|b - Ax\|_2^2 + \lambda \|x\|_2^2$$

Regularization for Ill-Posed Inverse Problems

Solutions to these problems usually formulated as:

$$\min_x \mathcal{L}(x) + \lambda \mathcal{R}(x)$$

where

- $\mathcal{L}$ is a goodness of fit function (e.g., negative log likelihood)
- $\mathcal{R}$ is regularization (reconstruct high and damp low frequencies)
- λ is regularization parameter

Example: General Tikhonov regularization:

$$\min_x \|b - Ax\|_2^2 + \lambda \|L\, x\|_2^2$$

Regularization for Ill-Posed Inverse Problems

Solutions to these problems usually formulated as:

$$\min_x \mathcal{L}(x) + \lambda \mathcal{R}(x)$$

where

- $\mathcal{L}$ is a goodness of fit function (e.g., negative log likelihood)
- $\mathcal{R}$ is regularization (reconstruct high and damp low frequencies)
- λ is regularization parameter

Example: General Tikhonov $\Leftrightarrow$ preconditioned standard Tikhonov:

$$\min_x \|b - AL^{-1}Lx\|_2^2 + \lambda \|Lx\|_2^2 \quad \Leftrightarrow \quad \begin{array}{c} \min_{\tilde{x}} \|b - \tilde{A}\,\tilde{x}\|_2^2 + \lambda \|\tilde{x}\|_2^2 \\ \tilde{A} = AL^{-1}, \quad \tilde{x} = L\,x \end{array}$$

Preconditioning for Ill-Posed Inverse Problems

Purpose of preconditioning:

- *not* to improve the condition number of the iteration matrix
- instead, preconditioning ensures the iteration vector lies in the "correct" subspace

P. C. Hansen.
Rank-deficient and discrete ill-posed problems.
SIAM, 1997.

M. Hanke and P. C. Hansen.
Regularization methods for large-scale problems.
Surv. Math. Ind., 3 (1993), pp. 253–315.

Preconditioning for Ill-Posed Inverse Problems

Purpose of preconditioning:

- *not* to improve the condition number of the iteration matrix
- instead, preconditioning ensures the iteration vector lies in the "correct" subspace

P. C. Hansen.
Rank-deficient and discrete ill-posed problems.
SIAM, 1997.

M. Hanke and P. C. Hansen.
Regularization methods for large-scale problems.
Surv. Math. Ind., 3 (1993), pp. 253–315.

Question: How to extend ideas to more general/complicated regularization?

Other Regularization Methods

In this talk we focus on solving

$$\min_x \|b - Ax\|_2^2 + \lambda \mathcal{R}(x)$$

where

- $\mathcal{R}(x) = \|x\|_p^p = \sum |x_i|^p \,, \quad p \geq 1$

  For example,

  - $p = 2$ is standard Tikhonov regularization
  - $p = 1$ enforces sparsity

  or

- $\mathcal{R}(x) = \left\| \sqrt{(D_h x)^2 + (D_v x)^2} \right\|_1 \quad$ (Total Variation)

Many Previous Works ...

Z. Wen, W. Yin, D. Goldfarb, and Y. Zhang.
A Fast Algorithm for Sparse Reconstruction Based on Shrinkage, Subspace Optimization, and Continuation.
SIAM J. Sci. Comput., 32 (2010), pp. 18321857.

R.H. Chan, Y. Dong, and M. Hintermuller.
An Efficient Two-Phase L^1-TV Method for Restoring Blurred Images with Impulse Noise.
IEEE Trans. on Image Processing, 19 (2010), pp. 1731–1739.

H. Fu, M.K. Ng, M. Nikolova and J.L. Barlow.
Efficient Minimization Methods of Mixed $\ell2$-$\ell1$ and $\ell1$-$\ell1$ Norms for Image Restoration.
SIAM J. Sci. Comput., 27 (2006), pp. 1881–1902.

Y. Huang, M.K. Ng, and Y.-W. Wen.
A Fast Total Variation Minimization Method for Image Restoration.
Multiscale Modeling and Simulation., 7 (2008), pp. 774–795.

T.F. Chan and S. Esedoglu.
Aspects of Total Variation Regularized L^1 Function Approximation.
SIAM J. Appl. Math., 65 (2005), pp. 1817–1837.

A. Borghi, J. Darbon, S. Peyronnet, T.F. Chan, and S. Osher.
A Simple Compressive Sensing Algorithm for Parallel Many-Core Architectures.
J. Signal Proc. Systems., 71 (2013), pp. 1–20.

Related Previous Work

S. Becker, J. Bobin, and E. Candès.
NESTA: A Fast and Accurate First-Order Method for Sparse Recovery.
SIAM J. Imaging Sciences, 4(1):1–39, 2011.

J.M. Bioucas-Dias and M.A.T. Figueiredo.
A new TwIST: two step iterative shrinkage/thresholding algorithms for image restoration.
IEEE Trans. Image Proc.,, 16 (2007), pp. 2992–3004.

S. Kim, K. Koh, M. Lustig, S. Boyd, and D. Gorinvesky.
An interior-point method for large-scale ℓ_1-regularized lest squares.
IEEE J. Selected Topics in Image Processing, 1 (2007), pp. 606–617.

J.P. Oliveira, J.M. Bioucas-Dias, M.A.T. Figueiredo.
Adaptive total vatiation image deblurring: A majorization-minimization approach.
Signal Processing, 89 (2009), pp. 1683–1693.

P. Rodríguez and B. Wohlberg.
An iteratively reweighted norm algorithm for total variation regularization.
In *Proceedings of the 40th Asilomar Conference on Signals, Systems and Computers (ACSSC)*, 2006.

S.J. Wright, R.D. Nowak, M.A.T. Figueiredo.
Sparse Reconstruction by Separable Approximation.
IEEE Transactions on Signal Processing, Vol. 57 No. 7 (2009), pp. 2479–2493.

Iteratively Reweighted Norm Approach (Wohlberg, Rodríguez)

- Iteratively construct L_m so that $\|L_m x\|_2^2 \approx \mathcal{R}(x)$, and compute

$$x_m = \arg\min_x \|b - Ax\|_2^2 + \lambda_m \|L_m x\|_2^2$$

Iteratively Reweighted Norm Approach (Wohlberg, Rodríguez)

- Iteratively construct L_m so that $\|L_m x\|_2^2 \approx \mathcal{R}(x)$, and compute

$$x_m = \arg\min_x \|b - Ax\|_2^2 + \lambda_m \|L_m x\|_2^2$$

- $\mathcal{R}(x) = \|x\|_1$

$$L_m = \mathrm{diag}\left(\frac{1}{\sqrt{|x_{m-1}|}}\right) = \texttt{diag(1 ./ sqrt(abs(}x_{m-1}\texttt{)))}$$

Iteratively Reweighted Norm Approach (Wohlberg, Rodríguez)

- Iteratively construct L_m so that $\|L_m x\|_2^2 \approx \mathcal{R}(x)$, and compute
$$x_m = \arg\min_x \|b - Ax\|_2^2 + \lambda_m \|L_m x\|_2^2$$

- $\mathcal{R}(x) = \|x\|_1$
$$L_m = \mathrm{diag}\left(\frac{1}{\sqrt{|x_{m-1}|}}\right) = \texttt{diag(1 ./ sqrt(abs(}x_{m-1}\texttt{)))}$$

- $\mathcal{R}(x) = TV(x) = \|\sqrt{(D_h x)^2 + (D_v x)^2}\|_1$,
$$L_m = S_m D_{hv}$$
where

Iterative Reweighted Norm Approach (Wohlberg, Rodríguez)

- Iteratively construct L_m so that $\|L_m x\|_2^2 \approx \mathcal{R}(x)$, and compute
$$x_m = \arg\min_x \|b - Ax\|_2^2 + \lambda_m \|L_m x\|_2^2$$

- $\mathcal{R}(x) = \|x\|_1$
$$L_m = \mathrm{diag}\left(\frac{1}{\sqrt{|x_{m-1}|}}\right) = \texttt{diag(1 ./ sqrt(abs(}x_{m-1}\texttt{)))}$$

- $\mathcal{R}(x) = TV(x) = \|\sqrt{(D_h x)^2 + (D_v x)^2}\|_1$,

where
$$L_m = S_m D_{hv}$$
$$D = \begin{pmatrix} 1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & -1 \end{pmatrix} \in \mathbb{R}^{(n-1)\times n}, \quad D_{hv} = \begin{pmatrix} D_h \\ D_v \end{pmatrix} = \begin{pmatrix} D \otimes I_n \\ I_n \otimes D \end{pmatrix}$$
$$\widetilde{x}_{m-1} = D_{hv} x_{m-1}, \quad \widetilde{S}_m = \mathrm{diag}\left(\frac{1}{\sqrt[4]{\sum_{i=1}^{2(N-n)} (\widetilde{x}_{m-1})_i}}\right), \quad S_m = \begin{pmatrix} \widetilde{S}_m & 0 \\ 0 & \widetilde{S}_m \end{pmatrix}$$

Krylov Subspace Methods for Tikhonov Regularization

Our approach: Similar to Wholberg and Rodriguez, combined with ideas in:

D. Calvetti, S. Morigi, L. Reichel, and F. Sgallari.
Tikhonov regularization and the L-curve for large discrete ill-posed problems.
J. Comput. Appl. Math., 123:423–446, 2000.

M. Hochstenbach and L. Reichel.
An iterative method for Tikhonov regularization with a general linear regularization operator.
J. Integral Equations Appl., 22:463–480, 2010.

L. Reichel, F. Sgallari, and Q. Ye.
Tikhonov regularization based on generalized Krylov subspace methods.
Appl. Numer. Math., 62:1215–1228, 2012.

S. Gazzola and P. Novati.
Automatic parameter setting for Arnoldi-Tikhonov methods.
Submitted.

Generalized Arnoldi-Tikhonov (GAT) Method

- Iteratively construct L_m so that $\|L_m x\|_2^2 \approx \mathcal{R}(x)$, and compute

$$x_m = \arg\min_x \|b - Ax\|_2^2 + \lambda_m \|L_m x\|_2^2$$

- Expensive if A is large, and many iteration steps (m) are needed.

Generalized Arnoldi-Tikhonov (GAT) Method

- Iteratively construct L_m so that $\|L_m x\|_2^2 \approx \mathcal{R}(x)$, and compute

$$x_m = \arg\min_x \|b - Ax\|_2^2 + \lambda_m \|L_m x\|_2^2$$

- Expensive if A is large, and many iteration steps (m) are needed.
- Our approach: Iteratively project problem onto Krylov subspace,

$$\mathcal{K}_m(A, b) = \text{span}\{b, Ab, \ldots, A^{m-1}b\}$$

Generalized Arnoldi-Tikhonov (GAT) Method

- Iteratively construct L_m so that $\|L_m x\|_2^2 \approx \mathcal{R}(x)$, and compute

$$x_m = \arg\min_{x \in \mathcal{R}^n} \|b - Ax\|_2^2 + \lambda_m \|L_m x\|_2^2$$

- Expensive if A is large, and many iteration steps (m) are needed.
- Our approach: Iteratively project problem onto Krylov subspace,

$$\mathcal{K}_m(A, b) = \text{span}\{b, Ab, \ldots, A^{m-1}b\}$$

- Get approximate solution by solving projected problem:

$$x_m = \arg\min_{x \in \mathcal{K}_m} \|b - Ax\|_2^2 + \lambda_m \|L_m x\|_2^2$$

Generalized Arnoldi-Tikhonov (GAT) Method

- Iteratively construct L_m so that $\|L_m x\|_2^2 \approx \mathcal{R}(x)$, and compute

$$x_m = \arg\min_{x \in \mathcal{R}^n} \|b - Ax\|_2^2 + \lambda_m \|L_m x\|_2^2$$

- Expensive if A is large, and many iteration steps (m) are needed.
- Our approach: Iteratively project problem onto Krylov subspace,

$$\mathcal{K}_m(A, b) = \text{span}\{b, Ab, \ldots, A^{m-1}b\}$$

- Get approximate solution by solving projected problem:

$$x_m = \arg\min_{x \in \mathcal{K}_m} \|b - Ax\|_2^2 + \lambda_m \|L_m x\|_2^2$$

  - Easier to solve projected problem.
  - As subspace grows (more iterations), get better approximations.

Generalized Arnoldi-Tikhonov (GAT) Method

- If $L_m = L$ remains constant, then to solve projected problem,

$$x_m = \arg\min_{x \in \mathcal{K}_m} \|b - Ax\|_2^2 + \lambda \|Lx\|_2^2$$

we need to construct an orthonormal basis $\{v_1, \ldots, v_m\}$ for $\mathcal{K}_m$.

Generalized Arnoldi-Tikhonov (GAT) Method

- If $L_m = L$ remains constant, then to solve projected problem,

$$x_m = \arg\min_{x \in \mathcal{K}_m} \|b - Ax\|_2^2 + \lambda \|Lx\|_2^2$$

  we need to construct an orthonormal basis $\{v_1, \ldots, v_m\}$ for $\mathcal{K}_m$.
- This can be done by the *Arnoldi Algorithm*, which computes:
  - $V_m = [v_1 \ \cdots \ v_m], \quad v_1 = b/\|b\|_2$
  - H_m is upper Hessenberg
  - $AV_m = V_{m+1} H_m$

Generalized Arnoldi-Tikhonov (GAT) Method

- If $L_m = L$ remains constant, then to solve projected problem,

$$x_m = \arg\min_{x \in \mathcal{K}_m} \|b - Ax\|_2^2 + \lambda \|Lx\|_2^2$$

  we need to construct an orthonormal basis $\{v_1, \ldots, v_m\}$ for $\mathcal{K}_m$.
- This can be done by the *Arnoldi Algorithm*, which computes:
  - $V_m = [v_1 \ \cdots \ v_m], \quad v_1 = b/\|b\|_2$
  - H_m is upper Hessenberg
  - $AV_m = V_{m+1} H_m$
- $x_m \in \mathcal{K}_m \ \Rightarrow \ x_m = V_m y$
- So we now need to find y from

$$\min_y \|AV_m y - b\|_2^2 + \lambda_m \|LV_m y\|_2^2$$

Generalized Arnoldi-Tikhonov (GAT) Method

The Arnoldi algorithm gives

- orthogonal property of V_m,
- the relation $AV_m = V_{m+1}H_m$,
- and $v_1 = b/\|b\|_2 \;\Rightarrow\; b = \|b\|_2 V_m e_1$

Generalized Arnoldi-Tikhonov (GAT) Method

The Arnoldi algorithm gives

- orthogonal property of V_m,
- the relation $AV_m = V_{m+1}H_m$,
- and $v_1 = b/\|b\|_2 \;\Rightarrow\; b = \|b\|_2 V_m e_1$

$$
\begin{aligned}
y_m &= \arg\min_y \|AV_m y - b\|_2^2 + \lambda_m \|LV_m y\|_2^2 \\
&= \arg\min_y \|V_{m+1}H_m y - \|b\|_2 V_{m+1} e_1\|_2^2 + \lambda_m \|LV_m y\|_2^2 \\
&= \arg\min_y \|V_{m+1}\left(H_m y - \|b\|_2 e_1\right)\|_2^2 + \lambda_m \|LV_m y\|_2^2 \\
&= \arg\min_y \|H_m y - \hat{b}\|_2^2 + \lambda_m \|LV_m y\|_2^2 \\
&= \arg\min_y \left\| \begin{bmatrix} H_m \\ \sqrt{\lambda_m} LV_m \end{bmatrix} y - \begin{bmatrix} \hat{b} \\ 0 \end{bmatrix} \right\|_2^2
\end{aligned}
$$

Generalized Arnoldi-Tikhonov (GAT) Method

The Arnoldi algorithm gives

- orthogonal property of V_m,
- the relation $AV_m = V_{m+1}H_m$,
- and $v_1 = b/\|b\|_2 \;\Rightarrow\; b = \|b\|_2 V_m e_1$

$$
\begin{aligned}
y_m &= \arg\min_y \|AV_m y - b\|_2^2 + \lambda_m \|LV_m y\|_2^2 \\
&= \arg\min_y \|V_{m+1}H_m y - \|b\|_2 V_{m+1} e_1\|_2^2 + \lambda_m \|LV_m y\|_2^2 \\
&= \arg\min_y \|V_{m+1}\left(H_m y - \|b\|_2 e_1\right)\|_2^2 + \lambda_m \|LV_m y\|_2^2 \\
&= \arg\min_y \|H_m y - \hat{b}\|_2^2 + \lambda_m \|LV_m y\|_2^2 \\
&= \arg\min_y \left\| \begin{bmatrix} H_m \\ \sqrt{\lambda_m} LV_m \end{bmatrix} y - \begin{bmatrix} \hat{b} \\ 0 \end{bmatrix} \right\|_2^2
\end{aligned}
$$

λ_m can be estimated in a smart way – see work by Gazzola and Novati.

Modifying Krylov Subspace Projection Method

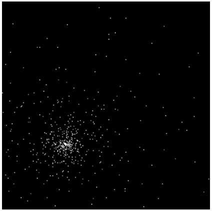
- Our previous explanation of projection method assumed $L_m = L$.
- That is, L did not change at each iteration.
- If L_m is changing at each iteration, need to use "Flexible" Krylov subspace methods; see, for example

  Y. Saad, Iterative Methods for Sparse Linear Systems, 2nd edition, SIAM, Philadelphia, 2003.

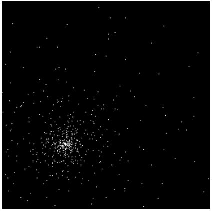
- Implementation details get tedious, so we skip these.

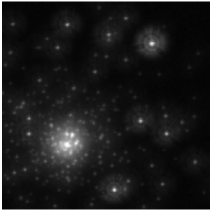
First Example - Star Cluster

First Example - Star Cluster

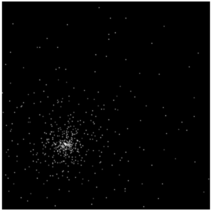
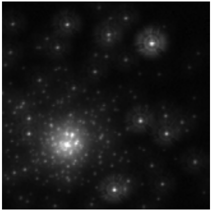
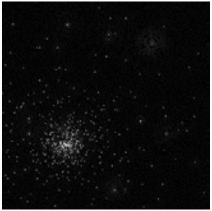


First Example - Star Cluster

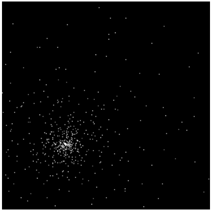
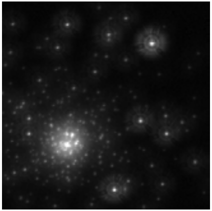
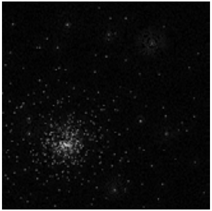




First Example - Star Cluster

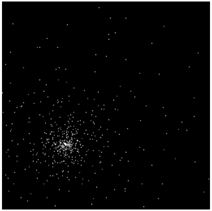




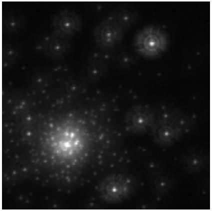
First Example - Star Cluster

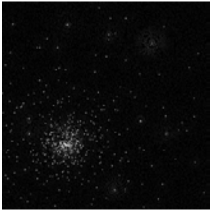




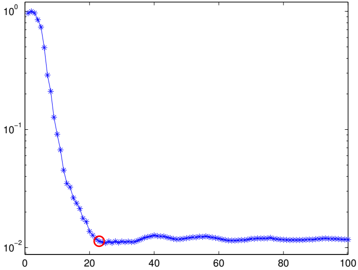
Stopping Iteration: 23 $\widetilde{\lambda} = 1.1976 \cdot 10^{-4}$.

First Example - Star Cluster



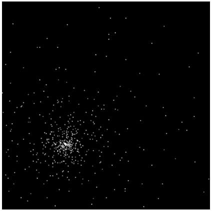


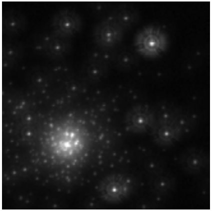


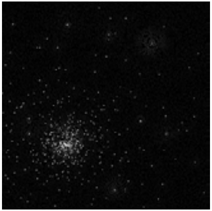


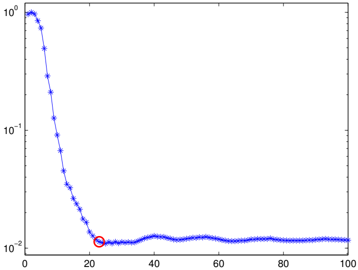
Stopping Iteration: 23 $\quad \widetilde{\lambda} = 1.1976 \cdot 10^{-4}$.

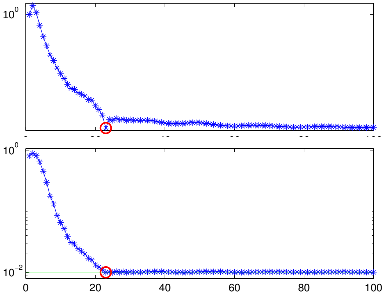
First Example - Star Cluster











Stopping Iteration: 23 $\widetilde{\lambda} = 1.1976 \cdot 10^{-4}$.

Restarting Strategy

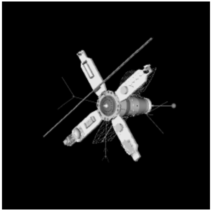
- For sparse reconstruction, L_m is diagonal $\Rightarrow$ it is easy to invert.
- In the Total Variation case,

$$L_m = S_m D_{hv}$$

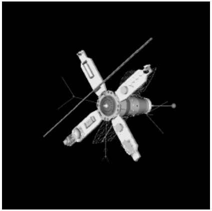
  is complicated, and not easy to invert.
- If L_m is not easy to invert, cost per iteration increases dramatically.
- So, we incorporate a restart strategy:
  - Restart when discrepancy principle is satisfied (residual reaches noise level).
  - Apply L_m at each restart.
  - Can also enforce nonnegativity with each restart.

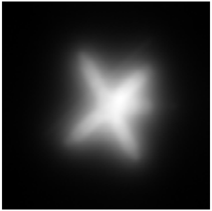
Second Example - Satellite

Second Example - Satellite

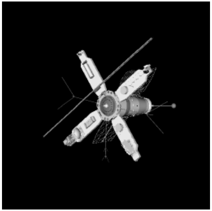


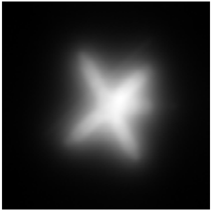
Second Example - Satellite

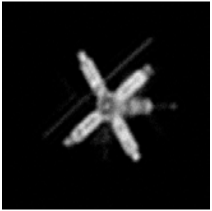




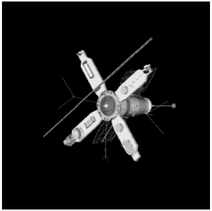
Second Example - Satellite

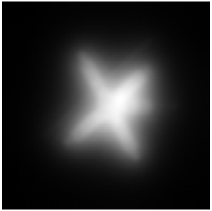


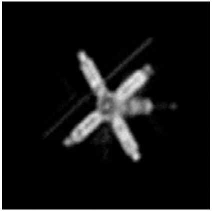




Second Example - Satellite

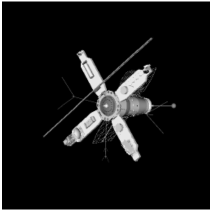


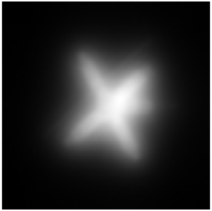


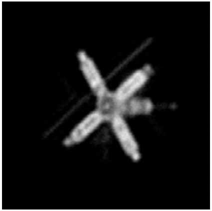


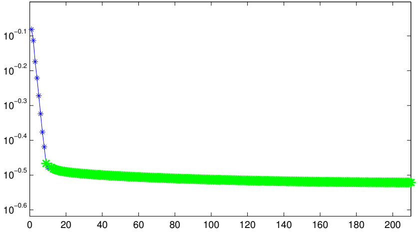
Outer Iterations: 200; Total Iterations: 212.

Second Example - Satellite





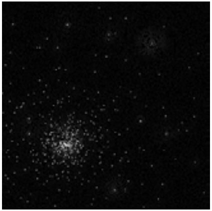


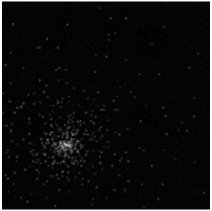


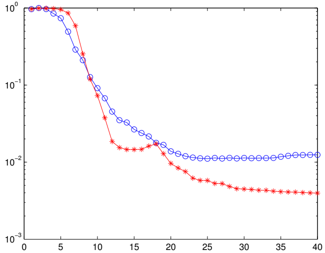
Outer Iterations: 200; Total Iterations: 212.

First Algorithm Revised

Including Flexible-AT approach into the Restarting-Nonnegative scheme

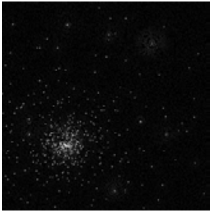


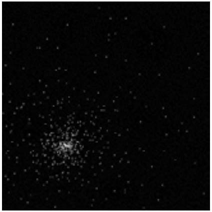


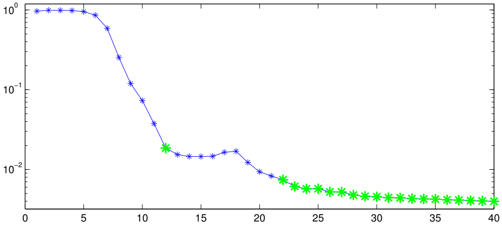


First Algorithm Revised

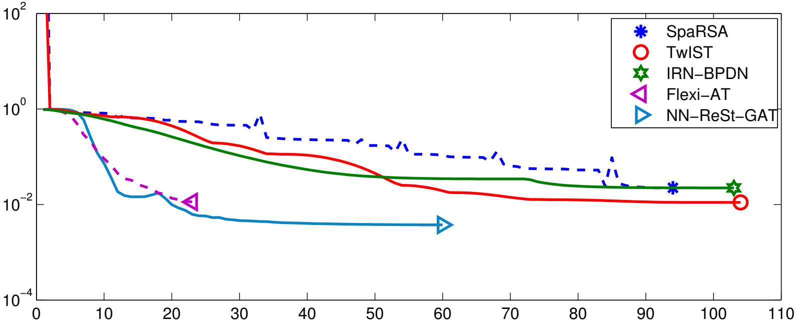
Including Flexible-AT approach into the Restarting-Nonnegative scheme







Comparison with other methods: Sparse Reconstructions



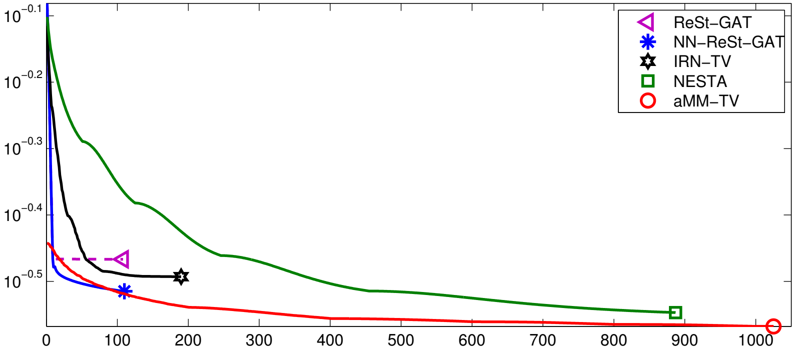
AT:	Standard Tikhonov regularization
SpaRSA:	Wright, Nowak, Figueiredo, 2007
TwIST:	Bioucas-Dias, Figueiredo, 2009
l1_ls:	Kim, Koh, Lustig, Boyd, Gorinvesky, 2007
IRN-BPDN:	Rodríguez, Wohlberg, 2009

Comparison with other method: Sparse Reconstructions

Method	Relative Error	Iterations	Total Time	Average Time
SpaRSA	$2.2365 \cdot 10^{-2}$	94	24.76	0.26
NESTA	$1.7800 \cdot 10^{-2}$	248	306.17	1.23
TwIST	$1.1089 \cdot 10^{-2}$	104	28.02	0.27
l1_ls	$2.2257 \cdot 10^{-2}$	298	683.55	2.29
IRN-BPDN	$2.2294 \cdot 10^{-2}$	103	35.72	0.35
AT	$1.8512 \cdot 10^{-2}$	12	0.91	0.08
RR-AT	$1.9171 \cdot 10^{-2}$	18	3.77	0.21
Flexi-AT	$1.1345 \cdot 10^{-2}$	23	2.44	0.11
ReSt-GAT	$1.1033 \cdot 10^{-2}$	51	5.95	0.12
NN-ReSt-GAT	$3.7530 \cdot 10^{-3}$	60	6.25	0.10

AT:	Standard Tikhonov regularization
SpaRSA:	Wright, Nowak, Figueiredo, 2007
NESTA:	Becker, Bobin, Candès, 2011
TwIST:	Bioucas-Dias, Figueiredo, 2009
l1_ls:	Kim, Koh, Lustig, Boyd, Gorinvesky, 2007
IRN-BPDN:	Rodríguez, Wohlberg, 2009

Comparison with other methods: TV Reconstructions



aMM-TV: Oliveira, Bioucas-Dias, Figueiredo, 2009
iRN-TV: Rodríguez, Wohlberg, 2006
NESTA: Becker, Bobin, Candès, 2011

Comparison with other method: TV Reconstructions

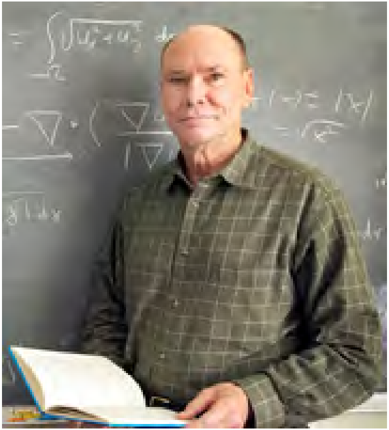
Method	Relative Error	Iterations	Total Time	Average Time
aMM-TV	$2.7056 \cdot 10^{-1}$	1025	2159.35	2.10
IRN-TV	$3.2141 \cdot 10^{-1}$	190	14.67	0.08
NESTA	$2.8382 \cdot 10^{-1}$	887	69.57	0.08
ReSt-GAT	$3.4138 \cdot 10^{-1}$	108	12.87	0.12
NN-ReSt-TV	$3.0556 \cdot 10^{-1}$	110	13.37	0.12
AT	$3.4176 \cdot 10^{-1}$	9	0.34	0.04
GAT	$3.4809 \cdot 10^{-1}$	9	0.70	0.08
RR-AT	$3.5321 \cdot 10^{-1}$	14	1.39	0.10

aMM-TV: Oliveira, Bioucas-Dias, Figueiredo, 2009
IRN-TV: Rodríguez, Wohlberg, 2006
NESTA: Becker, Bobin, Candès, 2011

Concluding Remarks

- Preconditioning (on the right) for ill-posed inverse problems:
  - Not used to improve condition number.
  - Used to regularize solution.
- Simple and efficient Krylov subspace methods for $\mathcal{R}(x) = \|Lx\|_2^2$ can be adapted to:
  - Sparse ($\| \cdot \|_1$) or TV regularization.
  - Requires flexible Krylov subspace framework.
  - Can incorporate regularization parameter choice methods and stopping criteria.
  - Restarting may be needed, but can be useful when enforcing projection constraints (e.g., nonnegativity).

A Bob Plemmons Story



Once upon a time, the computer was born ...

With the computer, then came ...

and the story continues, with many collaborators ...

and recognition by his peers ...

siam
Fellow

One notable time in this tale: 15 years ago

LINEAR ALGEBRA:
THEORY, APPLICATIONS, AND COMPUTATIONS

A Conference in Honor of
Robert J. Plemmons
On the Occasion of His 60th Birthday

One notable time in this tale: 15 years ago

LINEAR ALGEBRA:
THEORY, APPLICATIONS, AND COMPUTATIONS

A Conference in Honor of
Robert J. Plemmons
On the Occasion of His 60th Birthday

55 participants, including	
Avi Berman	Moody Chu
Mike Berry	Misha Kilmer
Raymond Chan	Jim Nagy
Tony Chan	Michael Ng

One notable time in this tale: 15 years ago

60th Birthday Conference program included the following:

> **... each of us has been greatly influenced not only by his scientific contributions, but also by his kindness and extreme generosity**.

One notable time in this tale: 15 years ago

60th Birthday Conference program included the following:

> **... each of us has been greatly influenced not only by his scientific contributions, but also by his kindness and extreme generosity**.
>
> **It is our pleasure to be part of this special event to honor our teacher, mentor, colleague and friend, Professor Robert J. Plemmons.**

One notable time in this tale: 15 years ago

60th Birthday Conference program included the following:

> **... each of us has been greatly influenced not only by his scientific contributions, but also by his kindness and extreme generosity**.
>
> **It is our pleasure to be part of this special event to honor our teacher, mentor, colleague and friend, Professor Robert J. Plemmons.**

Thanks to Raymond, Ronald and Michael for giving us an opportunity to once again express our gratitude, admiration, and deep respect for Bob!

And he lived happily ever after!